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Adding a club subset of w_2 without collapsing either w_1 or w_2

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ADDING A CLUB SUBSET OF ω_2
WITHOUT COLLAPSING EITHER ω_1 OR ω_2

A Thesis
Presented to
The Faculty of the Department of Mathematics
San José State University

In Partial Fulfillment
of the Requirements for the Degree
Master of Science

by
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ABSTRACT

ADDING A CLUB SUBSET OF ω_2 WITHOUT COLLAPSING EITHER ω_1 OR ω_2

by Ivan Zaigralin

The goal is to provide a characterization of sets which have club subsets in cardinal preserving generic extensions. Several results are to be presented and compared, the major cases being ω_1 and ω_2 preserving extensions, as well as different approaches to forcing. Where possible, a constructible forcing will be described.

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1. INTRODUCTION

Here we will bring together and relate several results concerning the characterization of sets which have club subsets in some ω_1 and ω_2 preserving extension. These results are:

- (1) If there exists a disjoint club sequence on $A \subseteq (\omega_2 \cap \text{cof } \omega_1)$, then A does not have a club subset in any ω_1 and ω_2 preserving outer model.
- (2) If $A \subseteq \omega_1$ is stationary, then A has a club subset in an ω_1 preserving outer model with no new ω -sequences of ordinals. If the CH holds in the inner model, then ω_2 is preserved in the outer model.
- (3) If $A \subseteq \omega_1$ is stationary, then A has a club subset in an ω_1 and ω_2 preserving outer model (even if CH is false in the inner model). New ω -sequences of ordinals are added.
- (4) If $A \subseteq \kappa$ is fat stationary, then under suitable cardinal arithmetic hypotheses, there exists an outer model in which A has a club subset and all cardinals are preserved.
- (5) If ω_2 is the L -successor of ω_1 and $D \subseteq \omega_2$ is very stationary, with a witness in L , then there exists a constructible forcing that adds a club subset to ω_2 and preserves all cardinals.

The two major cases we will have to deal with will be ω_1 and ω_2 case. As for the first one, our results will imply that the following statements are equivalent:

- (1) $A \subseteq \omega_1$ has a club subset in an ω_1 -preserving outer model.
- (2) $A \subseteq \omega_1$ has a club subset in a set forcing extension.
- (3) $A \subseteq \omega_1$ is stationary.

The first statement is not first-order, but the other two are.

The ω_2 case is different. The result due to Friedman (Theorem 6.1) is based on the following result.

Definition 1.1. Say that $A \subseteq \omega_2$ is *satisfiable* if A has a club subset in some ω_1 and ω_2 preserving outer model.

Theorem 1.2 (Anticharacterization).

- (1) Let L_{\min} be the minimum model of ZFC, that is, let L_{\min} be standard transitive universe satisfying $ZFC + V = L +$ “there are no standard models of ZFC”. Then

$$\{S \in L_{\min} \mid$$

$$S \subseteq \omega_2^{L_{\min}} \text{ is satisfiable with respect to outer models of } L_{\min}\} \notin L_{\min}.$$

- (2) If V is sufficiently non-minimal, then

$$\{S \in V \mid$$

$$S \subseteq \omega_2 \text{ is satisfiable with respect to outer models of } V\} \notin V.$$

- (3) Given any countable standard transitive model V of ZFC, there exists a definably class generic outer model V' of V such that

$$\{S \in V' \mid$$

$$S \subseteq \omega_2^{V'} \text{ is satisfiable with respect to outer models of } V'\} \notin V'.$$

Definably class generic here means that there exists a V -definable class forcing property P and a filter G on P meeting every dense class definable over V and such that $V' = V[G]$. *Sufficiently non-minimal* means that there exists a V -inaccessible cardinal κ and a Σ_2 -elementary $j : \text{Hyp}(V_\kappa) \rightarrow \text{Hyp}(V)$ with critical point κ and such that $j(V_\kappa) = V$. $\text{Hyp}(X)$ is the smallest admissible set with X as an element. The theorem appears in the work by M.C. Stanley [8].

This theorem implies that for models of ZFC, there cannot be a uniform, parameter free first-order equivalent to the second-order property “ $A \subseteq \omega_2$ has a club subset in some ω_1 and ω_2 preserving outer model.” Our motivation, then, is to say as much as we can about the satiable subsets of ω_2 , even though a first-order characterization relative to ZFC is impossible.

2. FORCING CONDITIONS AND GENERIC SETS

Here we will define the notion of a generic set and discuss how these can be added to transitive models to obtain generic extensions. Let us start by giving some definitions.

Definition 2.1. A set X is *transitive* whenever

$$\forall x \in X \forall y \in x (y \in X)$$

Definition 2.2. A set P together with a binary relation \leq is *partially ordered* (or a *poset*) if the order relation \leq is antisymmetric, reflexive, and transitive.

Let \mathfrak{M} be an arbitrary transitive model of ZFC, which we will call the *ground model*, and consider a non-empty poset $(P, <) \in M$. We will call P a *notion of forcing* and its elements—the *forcing conditions*. We will say that p is *stronger than* q iff $p < q$. We say that p and q are *compatible* whenever there exists r such that $r \leq p$ and $r \leq q$, *incompatible* otherwise. A subset W of P is *incompatible* when its elements are pairwise incompatible. We say that a subset D of P is *dense* if $\forall p \in P \exists d \in D (d \leq p)$.

Moreover, a subset F of P is a *filter on P* if the following conditions are satisfied:

- (1) $F \neq \emptyset$
- (2) $p \leq q \wedge p \in F \rightarrow q \in F$
- (3) $p, q \in F \rightarrow \exists r \in F (r \leq p \wedge r \leq q)$

A subset G of P is *generic over \mathfrak{M}* if

- (1) G is a filter on P
- (2) If D is dense in P and $D \in \mathfrak{M}$, then $G \cap D \neq \emptyset$

A subset D of P is *dense below p* if for every $p' \leq p$ there is $q \in D$ such that $q \leq p'$. A subset U of P is *open* if $p \in U \wedge q < p \rightarrow q \in U$. Finally, a subset C of P is *predense* if

every $p \in P$ is compatible with some $q \in C$. A subset W of P is an *antichain* in P if it is predense in P and if any two distinct $p, q \in W$ are incompatible.

Example 2.3. This example, due to Jech [1], will make it clear how adding a set leads to collapsing of ω_1 . Let \mathfrak{M} be the ground model. We want to add a set G to \mathfrak{M} to enlarge \mathfrak{M} in such a way that there exists a mapping in the extension $\mathfrak{M}[G]$ of ω onto $\omega_1^{\mathfrak{M}}$. (When such a mapping exists, we say that ω_1 has “collapsed”.) Suppose that a mapping exists with the required property, call it f . For each $n < \omega$ the initial segment $f \upharpoonright n$ is a finite sequence $(f(0), f(1), \dots, f(n-1))$, and it lies in \mathfrak{M} . Let P be the set of all finite sequences $(\alpha_0, \dots, \alpha_n)$ of countable ordinals. The set P will be our notion of forcing, and its elements—the forcing conditions.

Now consider two elements of P , p and q . If $p \subset q$, then, in a way, q gives us more information about f , and so we are justified in saying that q is *stronger* than p . By convention, we write it as $q < p$. That is,

$$q < p \iff p \subset q \tag{1}$$

We can see that this $<$ relation induces a partial ordering on P . Note also that both P and $(P, <)$ are in \mathfrak{M} .

Next, consider $G = \{f \upharpoonright n \mid n \in \omega\}$. $G \subset P$, and we can recover f from G . Namely, $f = \bigcup G$, and therefore G cannot be a set in \mathfrak{M} , on pain of having a surjective map from ω to ω_1 in \mathfrak{M} . In order for $\bigcup G$ to be a surjective map from ω to $\omega_1^{\mathfrak{M}}$, G must have the following properties:

- (1) If $p, q \in G$, then p and q are compatible, i.e. $p(k) = q(k)$ for all $k \in \text{dom } p \cap \text{dom } q$.
- (2) $\forall n \in \omega \exists p \in G (n \in \text{dom } p)$
- (3) $\forall \alpha \in \omega_1^{\mathfrak{M}} \exists p \in G (\alpha \in \text{ran } p)$

We now forget about f and let the set of conditions G be generic over \mathfrak{M} . Since G is a filter, $f = \bigcup G$ must be a function. We can also see that for all $n \in \omega$ the set

$$D_n = \{p \in P \mid n \in \text{dom } p\} \quad (2)$$

is dense in P . Similarly, for all $\alpha \in \omega_1^{\mathfrak{M}}$ the set

$$E_\alpha = \{p \in P \mid \alpha \in \text{ran } p\} \quad (3)$$

is dense in P . Therefore $G \cap D_n \neq \emptyset$ for all n and $G \cap E_\alpha \neq \emptyset$ for all α , and so $\text{dom } f = \omega$ while $\text{ran } f = \omega_1^{\mathfrak{M}}$. We can conclude that every transitive model \mathfrak{N} extending \mathfrak{M} and containing G satisfies that $\omega_1^{\mathfrak{M}}$ is a countable ordinal.

To generalize the previous example, we can prove a lemma, also appearing in Jech's work [1], which gives us a simple way of collapsing an arbitrary cardinal.

Definition 2.4. If κ is a regular cardinal, we will write $\lambda^{<\kappa}$ to denote $\sup\{\lambda^\alpha \mid \alpha < \kappa\}$

Lemma 2.5. Let κ be a regular cardinal and let $\lambda > \kappa$ be an arbitrary cardinal. There exists a notion of forcing $(P, <)$ such that it collapses λ onto κ , that is, λ has cardinality κ in the generic extension. Moreover,

- (1) every cardinal $\alpha \leq \kappa$ in the ground model \mathfrak{M} remains a cardinal in $\mathfrak{M}[G]$, and
- (2) if $\lambda^{<\kappa} = \lambda$, then every cardinal $\alpha > \lambda$ remains a cardinal in the extension.

Proof. Let P be the set of all functions p such that

- (1) $\text{dom } p \subseteq \kappa$ and $|\text{dom } p| < \kappa$,
- (2) $\text{ran } p \subseteq \lambda$,

and let $p < q$ iff $p \supset q$, as before. Let G be a generic filter on P and let $f = \bigcup G$. As in the example above, f is a function, and it maps κ onto λ . This is true because for all $\alpha < \kappa$ and all $\beta < \lambda$ the sets $D_\alpha = \{p \mid \alpha \in \text{dom } p\}$ and $E_\beta = \{p \mid \beta \in \text{ran } p\}$ are dense.

In order to show (1) and (2) we observe that $(P, <)$ is α -closed for every $\alpha < \kappa$, and therefore all cardinals $\leq \kappa$ are preserved; and if $\lambda^{<\kappa} = \lambda$, then $|P| = \lambda^+$ and it follows that all cardinals above λ are preserved. \square

3. FORCING WITH FINITE CONDITIONS: ω_1 CASE

There are two theorems in this section. Theorem (3.4) adds a club subset to a stationary subset of ω_1 using countable conditions. The proof is the precursor to theorems (4.3) and (4.4), which generalize it. This theorem is due to Baumgartner, Harrington, and Kleinberg [3] and independently to Jensen. The second theorem in this section, adding a club subset with finite conditions, is due to Baumgartner.

We start out by laying down some groundwork for discussing the preservation of ω_1 .

Definition 3.1. The set D is said to be *predense* below p if every condition below p is compatible with an element of D .

Lemma 3.2. Suppose that for every p and D_i , $i < \omega$, where each D_i is predense below p , there are $q \leq p$ and countable $d_i \subseteq D_i$, $i < \omega$, where d_i is predense below q for each $i < \omega$. Then ω_1 is preserved.

Proof. Assume that there is some p which forces σ to be a surjective map from ω to ω_1 . Then we can consider

$$D_i = \{q \mid \text{there exists } \alpha < \omega_1 \text{ such that } q \text{ forces } \sigma(i) = \alpha\} \quad (4)$$

D_i is dense, and hence predense. Let $d_i \subseteq D_i$ and $q \leq p$ be such that d_i is countable and predense below p . Let

$$\beta_i = \sup\{\alpha \mid \exists r \in d_i (r \Vdash \sigma(i) = \alpha)\}$$

Then $\beta_i < \omega_1$. Set $\beta = \sup_{i < \omega} \beta_i$. Then $\beta < \omega_1$ and $q \Vdash \sigma(i) < \beta$, for all $i < \omega$. It follows that there is $q \leq p$ and a countable β such that q forces $\sigma(i) < \beta$ for each $i < \omega$, that is, q forces that σ is bounded, which is a contradiction. \square

Now consider a theorem which appears in Baumgartner [3].

Definition 3.3. A set $S \subseteq \mathcal{P}_{\omega_1}(X)$ is *stationary* if for each club $C \subseteq \mathcal{P}_{\omega_1}(X)$ the intersection $S \cap C$ is nonempty.

Theorem 3.4. Let X be a subset of ω_1 . Then the following conditions are equivalent:

- (1) There exists an outer model which preserves ω_1 and X has a club subset in that model.
- (2) X is stationary.

Proof. (1) implies (2) because any two club sets have a non-empty intersection. Conversely, consider the forcing P with its conditions being closed, countable subsets of X , ordered by end-extension. Then P adds a club subset to X . We must show that ω_1 is preserved. To that end, suppose that $\langle D_i \mid i < \omega \rangle$ is a sequence of sets, each D_i predense below p . Choose a continuous elementary chain $\langle M_j \mid j < \omega_1 \rangle$, where each M_j is a countable elementary submodel of H_θ , θ being sufficiently large and regular, so that X , p , and $\langle D_i \mid i < \omega \rangle$ belong to M_0 and $M_j \in M_{j+1}$ for each j . Since $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is club, we can choose j in such a way that $\alpha = M_j \cap \omega_1 \in X$. Moreover, since each $D_i \cap M_j$ is predense below p on $P \cap M_j$, we can choose $p = p_0 \geq p_1 \geq p_2 \geq \dots$ so that $p_{i+1} \in M_j$ and it extends some $r_i \in D_i$ for each $i < \omega$, while at the same time $\sup(\bigcup_i p_i) = \alpha$. Then $q = \bigcup_i p_i \cup \{\alpha\}$ is a condition which extends p , and for each i , $d_i = D_i \cap M_j$ is predense below q . That gives us the statement of lemma (3.2), which implies that ω_1 is preserved. \square

At that point, Baumgartner asked a question: if X is a constructible, stationary subset of ω_1 , then is there a constructible forcing P which preserves ω_1 and adds a club subset to X ? The following theorem, due to Baumgartner, answers this question. (The proof also appears in Friedman's paper [6].)

Theorem 3.5. Let X be a stationary subset of ω_1 . Then there exists a forcing P which adds a club subset to X , such that P preserves ω_1 and P belongs to $L[X]$.

Proof. We can add a club subset to X with finite conditions, using the technique described by Uri Abraham [5]. A condition is a finite set p of disjoint closed intervals in ω_1 , whose left endpoints belong to X . Thus we allow the one-point intervals $[\alpha, \alpha]$, $\alpha \in X$. As we did before, we say that a condition q extends p iff $q \supseteq p$.

For a generic G , we say that C_G is the set of all left endpoints of intervals in $\bigcup G$. C_G must be an unbounded subset of X . Each countable ordinal either is a member of some interval in G or fails to be a limit point in X . It follows that C_G is closed in X . It remains to show that ω_1 is preserved.

So suppose that p is a condition and D_i , $i < \omega$ are predense below p . Choose a continuous elementary chain $\langle M_j \mid j < \omega_1 \rangle$ of countable elementary submodels of H_θ , θ being large and regular, so that X , p , and $\langle D_i \mid i < \omega \rangle$ all belong to M_0 and $M_j \in M_{j+1}$ for all j . Since $C = \{M_j \cap \omega_1 \mid j < \omega_1\}$ is club, we can choose j in such a way that $\alpha = M_j \cap \omega_1$ is in X . Let q be the condition $p \cup \{[\alpha, \alpha]\}$. If r extends q and $r_0 = r \restriction \alpha$ then every extension s_0 of r_0 in $P \cap M_j$ is compatible with r , since $[\alpha, \alpha] \in q$. It follows that $d_i = D_i \cap M_j$ is predense below q for every i , because if $r < q$ then we can choose $s_0 \leq r_0$ such that it extends a condition in d_i . Now, since s_0 is compatible with r , r must be compatible with some element of d_i . Hence ω_1 is preserved. \square

4. FORCING CLUB SETS

In this section we will consider some positive results due to Abraham and Shelah [5]. Specifically, we will see that given an arbitrary uncountable cardinal κ and a stationary $S \subseteq \kappa$, we may be able to find a generic extension which adds a club subset to S without collapsing any cardinals $\leq \kappa$. The sufficient condition for such an extension to be found is that S has to be fat. Moreover, the proofs of the following two theorems only work if GCH is assumed; for if $2^{\aleph_0} > \aleph_1$, then our forcing posets will cause \aleph_2 to collapse. The following theorem, due to Abraham, establishes the first interesting result.

Definition 4.1. A stationary set $S \subseteq \kappa$ is *fat* if for every club $C \subseteq \kappa$, $S \cap C$ contains closed subsets with arbitrarily large order types less than κ .

Definition 4.2. A cardinal κ is said to be *strongly inaccessible* if it is an uncountable, regular strong limit cardinal. We say that κ is a *strong limit* cardinal if it cannot be obtained through the power-set operation, i.e., for all $\lambda < \kappa$, $2^\lambda < \kappa$.

Theorem 4.3. Let κ be either a strongly inaccessible cardinal or the successor of a regular cardinal μ such that $\mu = \mu^{<\mu}$, and let $S \subseteq \kappa$ be fat. Then there is a poset P such that

- (1) Forcing with P adds a club $C \subseteq S$.
- (2) Forcing with P preserves all cardinals $\leq \kappa$.
- (3) Cardinality of P is $2^{<\kappa}$, and hence if $2^{<\kappa} = \kappa$, then all cardinals are preserved in the extension.

Proof. For a given fat $S \subseteq \kappa$ we define $P = \{p \mid p \subseteq S \text{ is a closed and bounded set of ordinals}\}$. P is partially ordered by end-extensions, namely,

$$p \leq p' \iff p = p' \cap (\sup(p) + 1)$$

The cardinality of P is $2^{<\kappa}$, and if G is a V -generic filter over P then $C = \bigcup\{p \mid p \in G\}$ is a club subset of S . It remains to show that our forcing does not cause any new sets with cardinality $< \kappa$ to appear in $V[G]$. We must prove that, given a regular cardinal $\tau < \kappa$ and a sequence $D = \langle D_i \mid i \in \tau \rangle$ of dense open subsets of P , it holds that $\bigcap_{i < \tau} D_i$ is dense in P .

Fix p and choose λ so that $H(\lambda)$, which is the collection of all sets with hereditary cardinality $< \lambda$, contains P . Let $M = \langle H(\lambda), \in \rangle$. Define a sequence $\langle M_\alpha \mid \alpha < \kappa \rangle$ of elementary substructures of M as follows:

- (1) $P, p, D \in M_0$, and some fixed well-ordering of $|P|$ are in M_0 . In other words, M_0 contains the universe of P .
- (2) $\tau + 1 \in M_0$.
- (3) M_α is of cardinality $< \kappa$, and if $\alpha < \beta$ then $M_\alpha \subset M_\beta$ and for limit δ , $M_\delta = \bigcup_{\eta < \delta} M_\eta$.
- (4) The intersection of the universe of M_α with κ , $c_\alpha = M_\alpha \cap \kappa$ is an ordinal and $\langle c_\alpha \mid \alpha < \kappa \rangle$ is a continuous increasing sequence, cofinal in κ .

To define M_α , we recall that it was assumed that $\beta^{<\beta} < \kappa$ for all $\beta < \kappa$. Hence for $\beta < \alpha < \kappa$, M_α contains each subset of β of cardinality $< |\beta|$.

Observe that $E = \{\alpha \mid \alpha = c_\alpha\}$ is a club subset of κ . Since S is fat, $S \cap E$ contains a closed subset of order-type $\tau + 1$, call it A . Let $\alpha = \sup A$. It follows that $A \cap \xi \in M_\alpha$ for each $\xi < \alpha$. We can construct in M_α an increasing sequence in P , $\langle p_i \mid i < \tau \rangle$, of length τ , such that $p_{i+1} \in D_i \cap M_\alpha$. We begin with $p = p_0$. If $p_i \in P \cap M_\alpha$ is defined then p_{i+1} is the first member of D_i such that the ordinal interval $(\sup p_i, \sup p_{i+1})$ has a non-empty intersection with A . For a limit $\delta < \tau$ we let

$$p_\delta = \bigcup_{i < \delta} p_i \cup (\sup (\bigcup_{i < \delta} p_i))$$

Only a proper initial segment of A is used in the definition of p_δ , and hence $p_\delta \in M_\alpha$; and since $A \subseteq S$ is closed, $p_\delta \subseteq S$. Finally,

$$p_\tau = \bigcup_{i < \tau} p_i \cup \{\alpha\} \in \bigcap_{i < \tau} D_i$$

shows that $\bigcap_{i < \tau} D_i$ is dense in P . \square

Difficulties will arise if we try to adapt the proof of Theorem (4.3) to the case where $\kappa = \mu^+$, μ singular. E.g., consider a fat set $S \subseteq \aleph_\omega^+$. If we follow the proof of the theorem above and define structures M_α of cardinality \aleph_ω then we cannot conclude that M_α are closed under countable unions of $p \in P$, since $\aleph_\omega^{\aleph_0} > \aleph_\omega$. We encounter the same problem in the case when $\tau^{<\tau} \geq \kappa$ for some $\tau < \kappa$. There is, however, an interesting result for $\kappa = \mu^+$, where μ is a singular strong limit. The following theorem, also due to Abraham [5], provides an example for $\kappa = \aleph_\omega^+$.

Theorem 4.4. *Suppose that \aleph_ω is strong limit and $S \subseteq \aleph_\omega^+$ is fat. Then there is a forcing poset P of cardinality 2^{\aleph_ω} which adds a club subset to S without collapsing any cardinals $\leq \aleph_\omega$.*

Proof. Let S be fat, $S \subseteq \aleph_\omega^+$. Without loss of generality, assume that $S \cap \aleph_\omega = \emptyset$. As we did in the proof of Theorem (4.3), define P to be the set of all bounded closed subsets of S . As before, we must prove that no new sets of size $\leq \aleph_\omega$ are added after forcing with P . It suffices to show that for all $n \leq \omega$ the intersection of \aleph_n many dense open subsets in P is dense. Fix a sequence of dense open sets $\langle D_i \mid i < \aleph_n \rangle$, as well as a condition $p \in P$. We will find an extension of p in $\bigcap_{i < \aleph_n} D_i$. Let $F(x, y)$ be a function such that for $\aleph_\omega \leq \alpha < \aleph_\omega^+$ we have

$$K : \beta \mapsto F(\alpha, \beta), \quad \beta < \alpha$$

is a bijection, $K(\alpha) = \aleph_\omega$. Next, pick a sequence $\langle M_\alpha \mid \alpha < \aleph_\omega^+ \rangle$ of structures of cardinality \aleph_ω , just like in Theorem (4.3), but require also that $F \in M_0$ and $\aleph_\omega \subset M_0$. Let

$c_\alpha = M_\alpha \cap \aleph_\omega^+$ and let $C = \{\alpha \mid \alpha = c_\alpha\}$ (C is club). Suppose that $2^{\aleph_{n-1}} = \lambda < \aleph_\omega$; then we can use the fact that S is fat to obtain a closed $B \subset S \cap C$ of order-type λ^+ . Define a function $h : [B]^2 \rightarrow \omega$ in the following way: for all $a, b \in B$, $a < b$, let $h(a, b) = k$ iff k is the least integer such that $F(b, a) \in \aleph_k$. Using the partition relation (described in detail by Williams [9]),

$$(2^{\aleph_{n-1}})^+ \rightarrow (\aleph_n)_{\aleph_{n-1}}^2$$

find $A \subseteq B$ of order-type \aleph_n which is homogeneous for some color k . Let $\alpha = \sup A$. Now, construct an increasing sequence $\langle p_i \mid i < \aleph_n \rangle$, just like in the proof of Theorem (4.3). Since every ordinal which is a limit in A is a member of S , $p_\delta \in P$ for limit $\delta < \aleph_n$. To make sure that $p_\delta \in M_\alpha$, we will show that every bounded subset $X \subseteq A$ is in M_α . Pick $b \in A$, $b > x$ for all $x \in X$. Then $F(b, x) < \aleph_k$ for all $x \in X$. The set $\{F(b, x) \mid x \in X, x < b\}$ is a subset of \aleph_k , and so it must be also in M_α . It follows that $X \in M_\alpha$. The proof can now be completed just like that of Theorem (4.3). \square

Remark 4.5. In respect to a forcing which preserves the cardinals and adds a club subset to a fat $S \in \kappa$, not much can be said if the GCH is not assumed. Baumgartner [3] provides a positive answer for $\kappa = \aleph_1$, but for $\kappa = \aleph_2$ the question is still unanswered.

5. DISJOINT CLUB SEQUENCES AND FAT STATIONARY SETS

In this section we will explore a more refined result due to Friedman and Krueger [2], which is related to the existence of disjoint club sequences. First, we need more definitions. For a set X which contains ω_1 , $\mathcal{F}_{\omega_1}(X)$ will denote the collection of countable subsets of X .

Definition 5.1. A subset C of $\mathcal{F}_{\omega_1}(X)$ is *cofinal* if, given any $x \in \mathcal{F}_{\omega_1}(X)$ there exists $z \in C$ such that $x \subseteq z$.

Definition 5.2. A set $C \subseteq \mathcal{F}_{\omega_1}(X)$ is *club* if it is closed under unions of countable increasing sequences and is cofinal in $\mathcal{F}_{\omega_1}(X)$.

Definition 5.3. If κ is a regular cardinal let $\text{cof } \kappa$ (and $\text{cof}(< \kappa)$ respectively) denote the class of ordinals with cofinality κ (and cofinality less than κ respectively).

If κ is a successor of a regular uncountable cardinal μ , this is equivalent to the statement that for every club $C \subseteq \kappa$, $S \cap C$ contains a closed subset with order type $\mu + 1$. In particular, if $A \subseteq \kappa^+ \cap \text{cof } \mu$ is stationary then $A \cup \text{cof}(< \mu)$ is fat.

Definition 5.4. If V is a transitive model of ZFC, we will say that W is an *outer model* of V if W is a transitive model of ZFC such that $V \subseteq W$ and W has the same ordinals as V . Moreover, $(W, V) \models \text{ZFC}$, i.e. W satisfies instances of replacement and separation that are formulated in a language with a predicate symbol in the inner model V .

Definition 5.5. Let T be a cofinal subset of $\mathcal{F}_{\omega_1}(\omega_2)$. We say that T is *thin* if for all $\beta < \omega_2$ the set $\{a \cap \beta \mid a \in T\}$ has size less than ω_2 .

In particular, if CH holds, then $\mathcal{F}_{\omega_1}(\omega_2)$ itself is thin.

Definition 5.6 (Martin's Maximum). MM is the statement that whenever P is a forcing poset which preserves stationary posets of ω_1 , then for any collection D of dense subsets of P with $|D| \leq \omega_1$, there is a filter $G \subset P$ which intersects each dense set in D .

We now introduce a combinatorial property of ω_2 which implies that there does not exist a thin stationary subset of $\mathcal{F}_{\omega_1}(\omega_2)$. This property follows from Martin's Maximum and is equiconsistent with Mahlo cardinal. The property will imply that there exists a fat stationary subset of ω_2 which cannot acquire a club subset by any forcing poset which preserves ω_1 and ω_2 . To be sure, we will not give a proof of the consistency of the existence of a disjoint club sequence; rather, we are going to show that if there exists a disjoint club sequence on $A \subseteq \omega_2 \cap \text{cof } \omega_1$, then A does not have a club subset in any ω_1 or ω_2 -preserving outer model.

Definition 5.7. A *disjoint club sequence* on ω_2 is a sequence $\langle C_\alpha \mid \alpha \in A \rangle$ such that A is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$, each C_α is a club subset of $\mathcal{F}_{\omega_1}(\alpha)$, and $C_\alpha \cap C_\beta$ is empty for all $\alpha < \beta$ in A .

Definition 5.8. A subset $S \subseteq \mathcal{F}_{\omega_1}(\omega_2)$ is *local club* if there is a club set $C \subseteq \omega_2$ such that for all uncountable $\alpha \in C$, $S \cap \mathcal{F}_{\omega_1}(\alpha)$ contains a club subset in $\mathcal{F}_{\omega_1}(\alpha)$.

Note that local club sets are also stationary. To see that, let K be a club set in $\mathcal{F}_{\omega_1}(\omega_2)$; then there exists a club $K' \subseteq \omega_2$ such that if $\alpha \in K'$ has cofinality ω_1 , then $K \cap \mathcal{F}_{\omega_1}(\alpha)$ is club in $\mathcal{F}_{\omega_1}(\alpha)$. Now suppose that $S \subseteq \mathcal{F}_{\omega_1}(\omega_2)$ is local club. Let C be club in ω_2 with the property that $S \cap \mathcal{F}_{\omega_1}(\alpha)$ is club, for all $\alpha \in C$. Let $\alpha \in C \cap K'$ have cofinality ω_1 . Then

$$(S \cap \mathcal{F}_{\omega_1}(\alpha)) \cap (K \cap \mathcal{F}_{\omega_1}(\alpha)) \neq \emptyset,$$

since both of these sets are club in $\mathcal{F}_{\omega_1}(\alpha)$.

Lemma 5.9. Suppose that $S \subseteq \mathcal{F}_{\omega_1}(\omega_2)$ is local club. Then S is local club in any outer model W with the same ω_1 and ω_2 .

Proof. We let C be local club subset of ω_2 , i.e. for every uncountable $\alpha \in C$, $S \cap \mathcal{F}_{\omega_1}(\alpha)$ contains a club subset in $\mathcal{F}_{\omega_1}(\alpha)$. Then C is also club in W . For each uncountable $\alpha \in C$ we fix a bijective map $g_\alpha : \omega_1 \rightarrow \alpha$. Then $\{g_\alpha \text{ ``} i \mid i < \omega_1 \}$ is a club subset of $\mathcal{F}_{\omega_1}(\alpha)$. If we intersect that set with S , we get a club subset of $S \cap \mathcal{F}_{\omega_1}(\alpha)$, namely, $\{a_i^\alpha \mid i < \omega_1\}$ which is increasing and continuous. This latter set remains a club subset of $\mathcal{F}_{\omega_1}(\alpha)$ in W . \square

Lemma 5.10. Suppose there is a disjoint club sequence $\langle C_\alpha \mid \alpha \in A \rangle$ on ω_2 . Let W be an outer model with the same ω_1 and ω_2 in which A is still stationary. Then there is a disjoint club sequence $\langle D_\alpha \mid \alpha \in A \rangle$ in W .

Proof. By the proof of lemma (5.9), each C_α contains a club set D_α in W . Since ω_1 is preserved, each α in A still has $\text{cof } \alpha = \omega_1$. \square

Theorem 5.11. Suppose that $\langle C_\alpha \mid \alpha \in A \rangle$ is a disjoint club sequence on ω_2 . Then $A \cup \text{cof } \omega$ does not contain a club subset.

Proof. For contradiction, suppose that $A \cup \text{cof } \omega$ does indeed contain a club subset. Without loss of generality we can assume that $2^{\omega_1} = \omega_2$ (otherwise we can work in a generic extension W by $\text{COLL}(\omega_2, 2^{\omega_1})$; in W the set $A \cap \text{cof } \omega$ contains a club subset and by lemma (5.10) there is a disjoint club sequence $\langle D_\alpha \mid \alpha \in A \rangle$).

Before we go on, we need some basic facts concerning the regressive functions. If $C \subseteq \mathcal{F}_{\omega_1}(X)$ is club then there exists a function $F : X^{<\omega} \rightarrow X$ such that every $a \in \mathcal{F}_{\omega_1}(X)$ which is closed under F is in C . If $F : X^{<\omega} \rightarrow \mathcal{F}_{\omega_1}(X)$ is a function and $Y \subseteq X$, we say that Y is *closed under F* for all $\gamma \in Y^{<\omega}$, where $F(\gamma) \subseteq Y$. A partial function $H : \mathcal{F}_{\omega_1}(X) \rightarrow X$ is *regressive* if for all $a \in \text{dom } H$ we have $H(a) \in a$.

Now, since $2^{\omega_1} = \omega_2$, $H(\omega_2)$ has the size ω_2 . Let h be a bijective map: $h : H(\omega_2) \rightarrow \omega_2$, and let \mathfrak{A} denote the structure $\langle H(\omega_2), \in, h \rangle$. Define B to be a set of all $\alpha \in$

$\omega_2 \cap \text{cof}(\omega_1)$ such that there exists an increasing and continuous sequence $\langle N_i \mid i < \omega_1 \rangle$ of countable elementary substructures of \mathfrak{A} such that

- (1) $N_i \in N_{i+1}$ for all $i < \omega_1$, and
- (2) the set $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is club in $\mathcal{F}_{\omega_1}(\alpha)$.

We will now show that B is stationary in ω_2 . Let $C \subseteq \omega_2$ be an arbitrary club set, and let \mathfrak{B} be the expansion of \mathfrak{A} by the function $\alpha \mapsto \min(C \setminus \alpha)$. Define by induction an increasing and continuous sequence $\langle N_i \mid i < \omega_1 \rangle$ of elementary substructures of \mathfrak{B} such that $N_i \in N_{i+1}$ for all $i < \omega_1$. Let $N = \bigcup \{N_i \mid i < \omega_1\}$. Then $\omega_1 \subseteq N$ and so $N \cap \omega_2$ is an ordinal. If we let $\alpha = N \cap \omega_2$, then $\alpha \in C$ and $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is club in $\mathcal{F}_{\omega_1}(\alpha)$. It is clear that $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is closed. To see that it is unbounded, let $x = \{\xi_n \mid n < \omega\}$ be an element of $\mathcal{F}_{\omega_1}(\alpha)$. In general $x \notin N$. However, for each n there exists $i_n < \omega_1$ such that $\xi_n \in N_{i_n}$. If $i = \sup_{n < \omega} i_n$, then $x \subseteq N_i$. So $\{N_i \cap \omega_2 \mid i < \omega_1\}$ is club and hence $\alpha \in B \cap C$.

Since $A \cup \text{cof } \omega$ contains a club subset, $A \cap B$ must be stationary. For each $\alpha \in A \cap B$ we fix a sequence $\langle N_i^\alpha \mid i < \omega_1 \rangle$, just as described in the definition of B above. It follows that $\{N_i^\alpha \cap \omega_2 \mid i < \omega_1\} \cap C_\alpha$ is club in $\mathcal{F}_{\omega_1}(\alpha)$, and so there must exist a club set $c_\alpha \subseteq \omega_1$ such that $\{N_i^\alpha \cap \omega_2 \mid i \in c_\alpha\}$ is club and is a subset of C_α . Let $i_\alpha = \min(c_\alpha)$ and let $d_\alpha = c_\alpha \setminus \{i_\alpha\}$.

Define $S = \{N_i^\alpha \mid \alpha \in A \cap B \wedge i \in d_\alpha\}$. If $N \in S$ then there is a unique pair $\alpha \in A \cap B$ and $i \in d_\alpha$ such that $N = N_i^\alpha$. To see that, assume that $N = N_i^\alpha = N_j^\beta$. It follows that $N \cap \omega_2 \in C_\alpha \cap C_\beta$, and so $\alpha = \beta$, which implies that $i = j$. Note that if $N_i^\alpha \in S$, then $N_{i_\alpha}^\alpha \in N_i^\alpha$. So the function $H : S \rightarrow H(\omega_2)$ defined by $H(N_i^\alpha) = N_{i_\alpha}^\alpha$ is well-defined and regressive.

We will prove that S is stationary in $\mathcal{F}_{\omega_1}(H(\omega_2))$. Let $F : H(\omega_2)^{<\omega} \rightarrow H(\omega_2)$ be a function. Recall that h is a bijection between $H(\omega_2)$ and ω_2 and define $G : \omega_2^{<\omega} \rightarrow \omega_2$

by letting

$$G(\alpha_0, \dots, \alpha_n) = h(F(h^{-1}(\alpha_0), \dots, h^{-1}(\alpha_n))) \quad (5)$$

Let E be a club subset of α in ω_2 close under G . Fix α in $E \cap A \cap B$. It follows that there exists $i \in d_\alpha$ such that $N_i^\alpha \cap \omega_2$ is closed under G . We claim that N_i^α is closed under F . Given $a_0, \dots, a_n \in N_i^\alpha$, the ordinals $h(a_0), \dots, h(a_n)$ are in $N_i^\alpha \cap \omega_2$. Define γ by letting

$$\gamma = G(h(a_0), \dots, h(a_n)) = h(F(a_0, \dots, a_n)) \quad (6)$$

We can see that $\gamma \in N_i^\alpha \cap \omega_2$, and therefore $h^{-1}(\gamma) = F(a_0, \dots, a_n) \in N_i^\alpha$.

Since S is stationary and $H : S \rightarrow H(\omega_2)$ is regressive, there exists a stationary set $S^* \subseteq S$ and a fixed N such that for all $N_i^\alpha \in S^*$ we have $H(N_i^\alpha) = N$. The set S^* is stationary, and hence its size must be ω_2 . It implies that there are distinct α and β such that for some $i \in d_\alpha$ and $j \in d_\beta$ both N_i^α and N_j^β are the members of S^* . Then $N = N_{i_\alpha}^\alpha = N_{j_\beta}^\beta$, and hence $N \cap \omega_2$ is in $C_\alpha \cap C_\beta$, which is a contradiction. \square

Lemma 5.12. Let $\langle C_\alpha \mid \alpha \in A \rangle$ be a disjoint club sequence and let W be an outer model of V with the same ω_1 and ω_2 . Then in W , $A \cup \text{cof } \omega$ does not contain a club subset.

Proof. If A remains stationary in W , then by lemma (5.10) there is a disjoint club sequence $\langle D_\alpha \mid \alpha \in A \rangle$ in W . By Theorem (5.11) $A \cap \text{cof } \omega$ does not contain a club subset in W . \square

Since it is our goal to characterize the cases when we can (or cannot) add a club subset without it resulting in a collapse, it makes sense to restate this result as follows:

Corollary 5.13. If $\langle C_\alpha \mid \alpha \in A \rangle$ is a disjoint club sequence in the inner model V of W , and $A \cap \text{cof } \omega$ contains a club subset in W , then any forcing of V which yields W will collapse either ω_1 or ω_2 .

6. FORCING WITH FINITE CONDITIONS: ω_2 CASE

Unfortunately, there is no analogous result for ω_2 . The following theorem is a direct consequence of the theorem due to Friedman [4].

Theorem 6.1. *Suppose that 0^\sharp exists. Then the set*

$$\{X \subseteq \omega_2^L \mid X \in L \text{ and}$$

$$X \text{ contains a club subset in an inner model where } \omega_2 = \omega_2^L\}$$

is not constructible, and has L -degree 0^\sharp .

E.g., there are some X which belong to the set above but have no club subset in any set-generic extension of L which preserves ω_2 .

There is, however, a sufficient condition (also due to Friedman [6]) for a subset of ω_2 to contain a club subset in an extension which is cardinal-preserving.

Definition 6.2. $X \subseteq \omega_2$ is *very stationary* if for all α in some stationary $X \subseteq X \cap \text{cof } \omega_1$, $X \cap \alpha$ contains a club subset of α .

With a variant of \diamond at ω_2 in L we can construct disjoint very stationary subsets of ω_2 in L . In general, $\diamond_\kappa(Q)$, where κ is a cardinal and $Q \subseteq \kappa$ is stationary, is the statement that there exists a sequence $\langle A_\alpha \mid \alpha \in Q \rangle$ such that

- (1) $A_\alpha \subseteq \alpha$ for each $\alpha \in Q$.
- (2) For each $A \subseteq \kappa$, the set $\{\alpha \in Q \mid A \cap \alpha = A_\alpha\}$ is stationary in κ .

The sufficient condition we have mentioned above is this:

Theorem 6.3. *If $X \subseteq \omega_2$ is very stationary then there is a set-forcing extension which preserves both ω_1 and ω_2 , and X contains a club subset in that extension.*

Proof. The proof is analogous to that of Theorem 3.4. We force with closed subsets of X of order type less than ω_2 , ordered by end-extension. We can use the fact that X is very stationary to show that if p is a condition and D_i , $i < \omega_1$, are predense below p then there exists $q \leq p$ which extends an element of D_i for each i (see the proof above for details). It follows that no new ω_1 -sequences are added by the forcing, which implies the preservation of ω_1 and ω_2 . \square

The most interesting question, however, is this: in the case of ω_2 , can we find a forcing similar to that of Baumgartner, which would add a club set with finite conditions? The following theorem will provide an answer.

Definition 6.4. D_0 witnesses that $D \subseteq \omega_2$ is very stationary if $D_0 \subseteq D \cap \text{cof } \omega_1$ is stationary and $D \cap \alpha$ contains a club subset of α for each $\alpha \in D_0$.

Theorem 6.5. Suppose that ω_2 is the L -successor to ω_1 , that $D \subseteq \omega_2$ is constructible, and that there exists a constructible witness that D is very stationary. Then there is a constructible forcing P which preserves cofinalities and adds a club subset of D .

Proof. Let $D_0 \in L$ witness that D is very stationary. We will assume that successor elements of D all have cofinality ω , by replacing D with $(D \cap \text{Lim } D) \cup \{\alpha + \omega \mid \alpha \in D\}$. This suffices because the set of limit points of a club set contained in this set is a club set contained in D .

Our condition will be a pair (A, S) , where

- (1) A is a finite set of disjoint closed intervals, their left endpoints being elements of D , and their left endpoints of $\text{cof } \omega_1$ being elements of D_0 . Let L denote the set of all left endpoints of intervals in A .
- (2) S is a finite set of countable, constructible Σ_1 elementary submodels x of some L_β , β limit, $\beta < \omega_2$, such that
 - (a) $x \cap \alpha$ is unbounded in $\alpha \in x$ whenever $\text{cof } \alpha = \omega$.

- (b) $\sup(x \cap \alpha) \in D$ whenever $\alpha \in (x \cap D_0); \omega_2$.
- (c) For $\alpha, \beta \in D$, if $\alpha < \beta$ are adjacent, $\alpha \in x$, $\beta < \sup(x \cap \text{Ord})$, then $\beta \in x$.
- (3) Given any interval $I = [\alpha, \beta] \in A$ and any $x \in S$ we have:
 - (a) If $I \cap x \neq \emptyset$ then $I \in x$.
 - (b) If $I \cap x = \emptyset$ and $\alpha < \sup(x \cap \text{Ord})$ then $\alpha_x \in L$, where α_x is the least ordinal $\geq \alpha$ in x .
- (4) Let F be the set of all elements of L of cofinality ω_1 , together with ω_2 . For nice x , we say that the F -height of x is the least element of F which is greater than $\sup(x \cap \text{Ord})$. As for our condition p , the following conditions must be met:
 - (a) If $x \in S$ and $\alpha \in F$ then $x \cap L_\alpha \in S$.
 - (b) If $x, y \in S$ have the same F -height, then $x = y$, $x \in y$, or $y \in x$.

From now on we will write (A_p, S_p) and F_p to designate the sets having to do with a condition p . We say that q extends p iff $A_q \supseteq A_p$ and $S_q \supseteq S_p$. Friedman breaks down the proof into three claims.

Claim 6.5.1. Fix $p \in P$.

- (1) For any club $C \subseteq \omega_2$ there is $\alpha \in C$, $\text{cof } \alpha = \omega_1$, such that $p \in L_\alpha$ and p^* , obtained by adding $[\alpha, \alpha]$ to A_p , is a condition extending p .
- (2) Let α and p^* be defined as above. If $q^* < p^*$ then there is $q < p$ in L_α such that every extension of q in L_α is compatible with q^* .

Claim 6.5.2. Fix $p \in P$.

- (1) For any club $C \subseteq \mathcal{F}_{\omega_1}(\omega_2)$ there is a constructible $x \in C$ such that $p \in x$ and p^* , defined by adding $x \cap L_\alpha$ to S_p for all $\alpha \in F_p$ is a condition extending p .
- (2) With q^* and p^* defined as above, if $q^* < p^*$ then there exists $q \in x$ which extends p , and every extension of q in x is compatible with q^* .

Claim 6.5.1 implies that ω_2 is preserved, while claim 6.5.2 implies that ω_1 is preserved. Since the size of P is ω_2 , all cofinalities are preserved. To complete the proof, we also need

Claim 6.5.3. Let G be P -generic and define

$$C_G = \{\gamma \mid \gamma \text{ is a left endpoint} \\ \text{of some interval in } \bigcup \{A_p \mid p \in G\}\}$$

Then C_G is a club subset of D .

Proof of claim 6.5.1.

Step 1. Choose α in the intersection of C and D_0 such that $p \in L_\alpha$. We must show that the properties (1) through (4) are satisfied. p^* satisfies (1) since α is greater than the right endpoint of any interval in A_p . Property (2) is the same for p^* as it is for p . Property (3a) is also the same for p^* , since α is not a member of any element of S_p . Ditto for (3b), since $\alpha > \sup x$ for all $x \in S_p$. Finally, property (4) is satisfied because $\alpha \notin x$ for all $x \in S_p$.

Step 2. Let q^* extend p^* and define q as follows:

$$A_q \text{ is } A_{q^*} \cap L_\alpha, \\ S_q \text{ is } S_{q^*} \cap L_\alpha$$

Step 2a. First of all, we will show that q is a condition. It suffices to verify properties (3b) and (4).

For property (3b), assume that $I \cap x$ is empty and the left endpoint β of $I = [\beta, \gamma]$ is $< \sup x$, where $I \in A_{q^*} \cap L_\alpha$ and $x \in S_{q^*} \cap L_\alpha$. Since q^* is a condition, β_x is the left endpoint of some interval J in S_{q^*} . But since $[\alpha, \alpha] \in A_{q^*}$, the right endpoint of J is $< \alpha$, and hence $J \in S_{q^*} \cap L_\alpha = S_q$.

For property (4), observe that $F_q = F_{q^*} \cap \alpha$, together with ω_2 . If $x \in S_q$ and $\beta \in F_q$ then $x \cap L_\beta \in S_{q^*}$, and therefore also in $S_q = S_{q^*} \cap L_\alpha$, because $x \cap L_\beta \in L_\alpha$. This gives us (4a). Now, if $x, y \in S_q$ have the same F_q -height then they also have the same F_{q^*} -height, since $x, y \in L_\alpha$. Moreover, q^* is a condition, and $x, y \in S_{q^*}$, which gives us (4b), and we can conclude that q is a condition.

Since q^* extends p^* , it also extends p . But $p \in L_\alpha$, and so q is a condition which extends p .

Step 2b. To complete the proof of this claim, we must also show that for any extension r of q , $r \in L_\alpha$, there exists a common extension t of r and q^* . Define t as follows:

$$\begin{aligned} A_t &= A_r \cup A_{q^*}, \\ S_t &= S_r \cup S_{q^*} \end{aligned}$$

It suffices to show that t is a condition, for in that case it will be clear that it extends both r and q^* . To finish the proof, we will verify that properties (1) through (4) hold for t .

For property (1), we observe that r is a condition extending q , each interval in A_r has its right endpoint $< \alpha$, and each interval $I \in A_{q^*} \setminus A_q$ has its left endpoint $\geq \alpha$. It follows that the intervals in A_t are disjoint.

Property (2) is evident; seeing (2c) uses the fact that D is unbounded in α , since $\alpha \in D_0$.

For property (3), we fix $I \in A_t \setminus A_r$ and $x \in S_r$. It follows that $\sup(x \cap \text{Ord}) < \alpha$ and the left endpoint of I is $\geq \alpha$. That is, property (3) is vacuously true. So fix $I \in A_r$ and $x \in S_t \setminus S_r$. Then $x \cap L_\alpha \in S_q \subseteq S_r$, implying that property (3) holds for I and $x \cap L_\alpha$. Since $I \cap x \neq \emptyset$ implies that $I \cap x \cap L_\alpha \neq \emptyset$, property (3a) holds for I and x . And so

does (3b). Really, if $I \cap x = \emptyset$ and the left endpoint β of I is $< \sup(x \cap \text{Ord})$ then I is disjoint from $x \cap L_\alpha$ and one of the following cases applies:

- (i) If $\beta < \sup(x \cap \alpha)$, then $\beta_x = \beta_{x \cap \alpha}$ and therefore (3b) follows, since r is a condition.
- (ii) If $\beta_x = \alpha$, then (3b) follows because $[\alpha, \alpha] \in A_{q^*}$.
- (iii) If $\beta_x = \alpha_x$, then we get (3b) because q^* is a condition.

The remaining cases, where $I \in A_r$ and $x \in S_r$, or $I \in A_t \setminus A_r$ and $x \in S_t \setminus S_r$, follow immediately from r and q^* being conditions.

For property (4a), it suffices to show that if $x \in S_t$ and $\beta \in F_t$ then $x \cap L_\beta \in S_t$. If $x \in S_r$ then either $\beta \in F_r$ or $\beta \geq \alpha$. If $\beta \in F_r$ then $x \cap L_\beta \in S_r \subseteq S_t$, since r is a condition. If, on the other hand, $\beta \geq \alpha$, then $x \cap L_\beta = x \in S_r \subseteq S_t$. On the other hand, if $x \in S_{q^*}$ then either $\beta \in F_{q^*}$ or $\beta \in F_r$. If $\beta \in F_{q^*}$ then (4a) follows since q^* is a condition. If $\beta \in F_r$ then $x \cap L_\beta = (x \cap L_\alpha) \cap L_\beta \in S_q \subseteq S_r$.

For property (4b), we must prove that if $x, y \in S_t$ have the same F_t -height, then $x \in y$, $y \in x$, or $x = y$. If $x \in S_r$ then F_t -height of x is at most α , implying that $y \in S_r$. Hence x and y have the same F_r -height and (4b) follows since r is a condition. If $x \in S_{q^*} \setminus S_r$ then the F_t -height of x is $> \alpha$, and hence $y \in S_{q^*}$. It follows that x and y have the same F_{q^*} -height, and (4b) follows since q^* is a condition.

This concludes the proof of claim 6.5.1.

The proof of the claim 6.5.2 follows the basic pattern of that of claim 6.5.1, but is somewhat more involved. We omit it here, but an interested reader may examine it in full in Friedman's paper [6].

Proof of the claim 6.5.3.

Here we must show that C_G is club. It is clearly unbounded, so it remains to show that it is also closed.

Let p be a condition. For contradiction, assume that

$$p \Vdash (\alpha \in \text{Lim } C_G \text{ and } \alpha \notin C_G)$$

We assume that for all $y \in S_p$, α_y is either a left endpoint of some interval in A_p , or it is forced by p that $\alpha_y \notin C_G$. For all $q < p$, $\alpha \notin I$ for all intervals $I \in A_q$, because in that case q forces either that $\alpha \in C_G$ or that α is not the limit of elements of C_G .

Suppose that $y \in S_p$, $\alpha > \sup(y \cap \alpha)$, and $\alpha < \sup(y \cap \text{Ord})$. Then α_y has to be a left endpoint of some $I \in A_p$, or else p forces that α is not a limit point of C_G .

Consider β , the least element of F_p which is $> \alpha$, and let

$$S = \{y \in S_p \mid \alpha \leq \sup(y \cap \text{Ord}) < \beta\}$$

It follows that the elements of S form an \in -chain.

Case 1. $y \cap \alpha$ is cofinal in α , for some $y \in S$. Assume that $y \cap \alpha$ is cofinal in α for some $y \in S$, and let y_0 be the \in -least element with that property. If α_{y_0} is a left endpoint of some interval $I \in A_p$, then $\alpha \in D$, as is required by the condition (2b) in the beginning of the proof. We will see that it is possible to extend p to force either that $\alpha \in C_G$ or that α is not a limit point of C_G . Before we do that, observe that $D \cap y_0 \cap \alpha$ is cofinal in α , since there are cofinally many $\gamma < \alpha$ which are forced by extensions of p into C_G , and for all such $\gamma \notin y_0$, $\gamma_{y_0} \in D_0 \subseteq D$. It follows from the property (2c) for y_0 that $D \cap y_0 \cap \alpha \cap \text{cof } \omega$ is also cofinal in α .

Subcase 1.1. α_{y_0} is not the left endpoint of some interval in A_p . If α_{y_0} is not the left endpoint of any interval in A_p , then we fix $\gamma \in D \cap y_0 \cap \alpha \cap \text{cof } \omega$ such that $\gamma > \delta$ for all δ , right endpoints of each interval in A_p such that its left endpoint is less than α and at the same time larger than $\sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$. We will show that a condition results when the interval $I = [\gamma, \alpha_{y_0}]$ is added to p . This condition will then force that $\alpha \notin \text{lim } C_G$ in contradiction to our initial hypothesis. By

our choice of γ , I is disjoint from each interval in A_p with its left endpoint $< \alpha$. And since we assumed that α_{y_0} is not the left endpoint of an interval in A_p , and so neither is any ordinal between α and α_{y_0} , I is also disjoint from each interval in A_p with its left endpoint $> \alpha$. By our choice of γ , I also has an empty intersection with all $y \in S_p$ such that $\sup(y \cap \beta) < \alpha$. Moreover, I is disjoint from every $y \in S_p$ with

$$\sup(y \cap \alpha) < \alpha \leq \sup(y \cap \beta),$$

since for such y we can write $\alpha_y > \alpha_{y_0}$, because α_y is the left endpoint of some interval in A_p while α_{y_0} is not. Every other $y \in S_p$ contains y_0 , and hence also I as its element. If $y \in S_p$ is disjoint from I with $\gamma < \sup(y \cap \text{Ord})$, then $\gamma_y = \alpha_y$ is a left endpoint of some interval in A_p .

Subcase 1.2. α_{y_0} is the left endpoint of some interval in A_p . If α_{y_0} is the left endpoint of some interval in A_p , then we set $I = [\alpha, \alpha]$. We will show that a condition results if we add I to p which will force that $\alpha \in C_G$, in contradiction to our initial hypothesis. Clearly, I is disjoint from each interval in A_p , since α is not in any such interval. If I intersects $y \in S_p$, then $I \in y$. If I is disjoint from $y \in S_p$ and $\alpha < \sup(y \cap \text{Ord})$, then $\alpha_y \geq \alpha_{y_0}$, on pain of having $y \cap L_\beta \in S$, $y_0 \in y \cap L_\beta$, and hence $\alpha = \sup(y_0 \cap \alpha_{y_0}) \in y$, which is against our hypothesis. So again, α_y must be the left endpoint of some interval in A_p , or else α_{y_0} cannot be such.

Case 2. $y \cap \alpha$ is not cofinal in α , for all $y \in S$. For this last case, choose $I = [\gamma, \alpha]$, $\gamma \in D \cap \alpha \cap \text{cof } \omega$ just as we did before, i.e.

- (1) $\gamma > \delta$ for all δ , right endpoints of each interval in A_p such that its left endpoint is $< \alpha$, and
- (2) $\gamma > \sup(y \cap \alpha)$ for all $y \in S_p$ with $\sup(y \cap \alpha) < \alpha$

We will show that a condition results when we add I to p . By our choice of γ , I is disjoint from all intervals in A_p . I is also disjoint from each $y \in S_p$, since $\sup(y \cap \alpha) < \gamma$

by our choice of γ and the hypothesis for this case. (If $\alpha \in \gamma$, then $\alpha = \gamma_y$ and hence p forces that α is not a limit of elements of C_G , since α is not the left endpoint of any interval in A_q for each $q \leq p$.) If $y \in S_p$ and $\gamma < \sup(y \cap Ord)$, then α_y , yet again, is the left endpoint of some interval in A_p .

The proof of this claim also concludes the proof of the theorem. □

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